# Midterm Exam - Analysis IV
# B. Math III

20 February, 2023

> (i) Duration of the exam is 3 hours.
>
> (ii) The maximum number of points you can score in the exam is 100 (total = 105).
>
> (iii) You are not allowed to consult any notes or external sources for the exam.

Name: ____________________

Roll Number: ____________________

1. (a) (15 points) Show that there exists a linear functional $Lim : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ with the following properties:
   (i) $Lim$ is a positive linear functional of norm one.
   (ii) For each sequence $x := \{x_n\} \in \ell^\infty(\mathbb{N})$, we have
   $$\liminf_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} \leq Lim(x) \leq \limsup_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n}.$$
   (iii) For each sequence $x := \{x_n\} \in \ell^\infty(\mathbb{N})$, we have
   $$Lim(x) = Lim(\tau_1(x)),$$
   where $\tau_1 : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ denotes the left-shift operator. In other words, $Lim$ is invariant under left-shifts of sequences.

(b) (10 points) Let $\varphi : [0,1] \to [0,1]$ be a continuous function and $\Phi : C([0,1]) \to C([0,1])$ be defined by $\Phi(f) = f \circ \varphi$.. Show that there is a $\Phi$-invariant linear functional $\rho : C([0,1]) \to \mathbb{R}$, that is, $\rho \circ \Phi = \rho$.

Total for Question 1: 25

2. (20 points) Let $f, g : [0,1] \to \mathbb{R}$ be two Lebesgue-integrable functions satisfying

$$\int_0^t f(x)\, dx \leq \int_0^t g(x)\, dx,$$

for all $t \in [0,1]$. If $\varphi : [0,1] \to \mathbb{R}$ is a non-negative decreasing function, show that the functions $\varphi f$ and $\varphi g$ are Lebesgue-integrable over $[0,1]$ and that they satisfy

$$\int_0^t \varphi(x) f(x)\, dx \leq \int_0^t \varphi(x) g(x)\, dx$$

for all $t \in [0,1]$.

Total for Question 2: 20

3. (15 points) Assume that $f$ is continuous on $[0,1]$, $f(0) = 0, f'(0)$ exists. Prove that the Lebesgue integral

$$\int_0^1 f(x) x^{-\frac{3}{2}}\, dx$$

exists.

Total for Question 3: 15

4. (a) (10 points) Show that $\log \frac{1}{1-x} \in L^1([0,1]; dx)$ and with justification, compute the following integral:

$$\int_0^1 \log \frac{1}{1-x}\, dx.$$

(b) (10 points) Prove that the function,

$$\frac{1}{1 + x^2 \sin^2 x},$$

is not Lebesgue-integrable on $[1, \infty)$.

Total for Question 4: 20

5. (a) (15 points) Let $F(y) = \int_0^\infty \frac{\sin xy}{x(x^2+1)}\, dx$ if $y > 0$. Show that $F$ satisfies the differential equation $F''(y) - F(y) + \frac{\pi}{2} = 0$ and deduce that $F(y) = \frac{1}{2}\pi(1 - e^{-y})$.

(b) (10 points) For $a > 0, y > 0$, compute the value of

$$\int_0^\infty \frac{\cos xy}{(x^2 + a^2)}\, dx,$$

with proper justification of the steps.

Total for Question 5: 25